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THE ABSOLUTE AND RELATIVE BETTI NUMBERS OF A MANIFOLD WITH BOUNDARY

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1. Consider a compact manifold M with a boundary B, so that M is the closure of an open submanifold of an n-dimensional orientable Riemannian manifold V, and B is a compact orientable $(n-1)$-dimensional manifold. Let $H_p(M, \boldsymbol{R})$ and $H_{n-p}\{(M, B), \boldsymbol{R}\}$ be respectively the pth Betti group of M and the pth Betti group of M(mod. B). Then by Lefschetz duality theorem the pth Betti group $H_p(M, \boldsymbol{R})$ and the $(n-p)$th Betti group $H_{n-p}\{(M, B), \boldsymbol{R}\}$ are dual, so that the absolute Betti number A_p and the relative Betti number R_{n-p} of the manifold M are equal. For a k-pinched manifold M, the numbers A_p and R_q for $p = q = 2$ are zero, when the number k is greater than a number λ and the second fundamental form on B satisfies some conditions. We can improve the number λ, when the dimension of the manifold M is 5. These results are a generalization of those given in [8].

2. If α, β are two tensors of the manifold M of order p, then the local inner product of the two tensors α, β is defined by

$$(\alpha, \beta) = \frac{1}{p!}\alpha^{i_1\cdots i_p}\beta_{i_1\cdots i_p} = \frac{1}{p!}\alpha_{i_1\cdots i_p}\beta^{i_1\cdots i_p} ,$$

and the local norm of the tensor α is defined by

$$|\alpha|^2 = \frac{1}{p!}\alpha^{i_1\cdots i_p}\alpha_{i_1\cdots i_p} .$$

If η is the volume element of the manifold M, then the global inner product of the two tensors α, β and the global norm of the tensor α are defined, respectively,

$$\langle\alpha, \beta\rangle = \int_M (\alpha, \beta)\eta , \qquad \|\alpha\|^2 = \int_M |\alpha|^2\eta .$$

If α is a p-form, then we have [6, p. 187]

$$\langle\Delta\alpha, \alpha\rangle = \|d\alpha\|^2 + \|\delta\alpha\|^2 , \tag{2.1}$$

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where $\Delta\alpha, d\alpha$ and $\delta\alpha$ are the Laplacian, the exterior differentiation and the codifferentiation of α given by [7, pp. 1–2]

$$\Delta\alpha = d\delta\alpha + \delta d\alpha \ , \tag{2.2}$$

$$(d\alpha)_{j_1\cdots j_{p+1}} = \frac{1}{p!}\varepsilon^{li_1\cdots i_p}_{j_1\cdots j_{p+1}}\nabla_l\alpha_{i_1\cdots i_p} \ , \tag{2.3}$$

$$(\delta\alpha)_{i_2\cdots i_p} = -\nabla_l\alpha^l{}_{i_2\cdots i_p} \ . \tag{2.4}$$

The following relation is also valid [7, p. 4]:

$$\frac{1}{2}\Delta(|\alpha|^2) = (\alpha, \Delta\alpha) - |\nabla\alpha|^2 + \frac{1}{2[(p-1)!]}Q_p(\alpha) \ , \tag{2.5}$$

where

$$|\nabla\alpha|^2 = \frac{1}{p!}\nabla_l\alpha_{i_1\cdots i_p}\nabla^l\alpha^{i_1\cdots i_p} \ , \tag{2.6}$$

$$Q_p(\alpha) = (p-1)R_{ijhl}\alpha^{iji_3\cdots i_p}\alpha^{hl}{}_{i_3\cdots i_p} - 2R_{hl}\alpha^{hi_2\cdots i_p}\alpha^l{}_{i_2\cdots i_p} \ . \tag{2.7}$$

For a point P on the boundary B, let $(u^1, \cdots, u^{n-1})$ and $(v^1, \cdots, v^n)$ be two local coordinate systems of the point P considered as a point of B and M respectively. Then the boundary B is represented locally by

$$v^i = f^i(u^1, \cdots, u^{n-1}) \ , \qquad i = 1, \cdots, n \ , \tag{2.8}$$

in $U(P) \cap M$, where $U(P)$ is a coordinate neighborhood of V. Denote by N the normal vector field of the boundary B and choose the coordinate system $(u^1, \cdots, u^{n-1})$ such that the vector fields $N, \partial/\partial u^1, \cdots, \partial/\partial u^{n-1}$ form a positive sense of M with respect to the basis $\partial/\partial u^1, \cdots, \partial/\partial u^n$. Then Stokes' theorem can be stated as follows. If $\gamma = (\gamma_i)$ is an arbitrary vector field on M, then [9, p. 589]

$$\int_B (\gamma, N)\bar{\eta} = -\int_M \delta\gamma\,\eta \ , \tag{2.9}$$

where

$$\bar{\eta} = \sqrt{h}\,du^1 \wedge \cdots \wedge du^{n-1}, \eta = \sqrt{g}\,dv^1 \wedge \cdots \wedge dv^n \ ,$$

h being the determinant of the metric on the boundary B, which is obtained under the assumption that the mapping F defined by (2.8) is an isometric immersion of M into B.

A p-form $\alpha = (\alpha_{i_1\cdots i_p})$ is tangential to B if it satisfies the relation [10, p. 431]:

$$\alpha^{i_1\cdots i_p} = (\partial v^{i_1}/\partial u^{j_1}) \cdots (\partial v^{i_p}/\partial u^{j_p})\bar{\alpha}^{j_1\cdots j_p} ,$$

or

$$\alpha^{ii_2\cdots i_p}N_i = 0 ,$$

on B, where $\bar{\alpha} = (\bar{\alpha}_{i_1\cdots i_p})$ is a p-form defined over B. The p-form α satisfies also the relation [10, p. 434]:

$$(2.10) \qquad \begin{aligned}(\nabla_l\alpha_{hi_2\cdots i_p})\alpha^{hi_2\cdots i_p}N^l = {} & pH_{ij}\bar{\alpha}^i{}_{i_2\cdots i_p}\alpha^{ji_2\cdots i_p} \\ & - (p+1)(\nabla_{[l}\alpha_{hi_2\cdots i_p]})\alpha^{li_2\cdots i_p}N^h ,\end{aligned}$$

where

$$\begin{aligned}(p+1)\nabla_{[l}\alpha_{hi_2\cdots i_p]} = {} & \nabla_l\alpha_{hi_2\cdots i_p} - \nabla_h\alpha_{li_2\cdots i_p} \\ & - \nabla_{i_2}\alpha_{hli_3\cdots i_p} - \cdots - \nabla_{i_p}\alpha_{hi_2\cdots i_{p-1}l} .\end{aligned}$$

A p-form $\alpha = (\alpha_{i_1\cdots i_p})$ on the manifold M is normal to the boundary B, if it satisfies the relation [10, p. 432]:

$$\alpha_{i_1\cdots i_p}(\partial v^{i_1}/\partial u^{j_1}) \cdots (\partial v^{i_p}/\partial u^{j_p}) = 0 ,$$

from which we obtain [10, p. 435]

$$(2.11) \qquad \begin{aligned}(\nabla_h\alpha_{li_2\cdots i_p})\alpha^{li_2\cdots i_p}N^h = {} & p(\nabla_h\alpha^h{}_{i_2\cdots i_p})\alpha^{li_2\cdots i_p}N_l \\ & + pH^l{}_l\bar{\alpha}_{i_2\cdots i_p}\bar{\alpha}^{i_2\cdots i_p} - (p-1)pH_{ij}\bar{\alpha}^i{}_{i_3\cdots i_p}\bar{\alpha}^{ji_3\cdots i_p} ,\end{aligned}$$

where $\bar{\alpha} = (\bar{\alpha}_{i_2\cdots i_p})$ is a $(p-1)$-form defined by

$$\alpha_{li_2\cdots i_p}N^l = \bar{\alpha}_{j_2\cdots j_p}(\partial v^{j_2}/\partial u^{i_2}) \cdots (\partial v^{j_p}/\partial u^{i_p}) .$$

3. Assume that the manifold M is of odd dimension $n = 2m + 1$ and admits a metric which is positively k-pinched, and let α be a harmonic 2-form on the manifold M. Then for any point P of the manifold there is a special basis $(X_1^*, \cdots, X_n^*)$ in the vector space M_P^* such that at the point P, α can be written as

$$(3.1) \quad \alpha = \alpha_{12}X_1^* \wedge X_2^* + \alpha_{34}X_3^* \wedge X_4^* + \cdots + \alpha_{2m-1,2m}X_{2m-1}^* \wedge X_{2m}^* .$$

Now consider the $2m$-form β defined by

$$(3.2) \qquad \beta = \frac{1}{m!}\alpha \wedge \cdots \wedge \alpha, \ (m \text{ times}) ,$$

which becomes, in consequence of (3.1),

$$(3.3) \qquad \beta = \alpha_{12}\alpha_{34} \cdots \alpha_{2m-1,2m}X_1^* \wedge \cdots \wedge X_{2m}^* .$$

Since the manifold M is k-pinched, $Q_2(\alpha)$ and $Q_{2m}(\beta)$ satisfy the inequalities [8]:

$$\frac{1}{2}Q_2(\alpha) \leq -2(2m-1)k|\alpha|^2 + \frac{8}{3}(1-k)\delta , \tag{3.4}$$

$$\frac{1}{2[(2m-1)!]}Q_{2m}(\beta) \leq -2mk|\beta|^2 , \tag{3.5}$$

where

$$\begin{gathered} |\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \cdots + \alpha_{2m-1,2m}^2, \qquad |\beta|^2 = \alpha_{12}^2\alpha_{34}^2\cdots\alpha_{2m-1,2m}^2 , \\ \delta = \alpha_{12}\alpha_{34} + \cdots + \alpha_{12}\alpha_{2m-1,2m} + \cdots + \alpha_{2m-3,2m-2}\alpha_{2m-1,2m} . \end{gathered} \tag{3.6}$$

Applying the Laplace operator Δ to the function $|\beta|^2$, we get $\Delta(|\beta|^2) = \delta d(|\beta|^2)$ and therefore

$$\int_M \Delta(|\beta|^2)\eta = \int_M \delta d(|\beta|^2)\eta .$$

By means of (2.5) and (2.9), the above relation becomes

$$-\frac{1}{2}\int_B (N, d(|\beta|^2))\bar{\eta} = \int_M \left[(\beta, \Delta\beta) - |\nabla\beta|^2 + \frac{1}{2[(2m-1)!]}Q_{2m}(\beta)\right]\eta ,$$

which takes the form, due to (2.1),

$$\frac{1}{2}\int_B (N, d(|\beta|^2))\bar{\eta} = \int_M \left[-|d\beta|^2 - |\delta\beta|^2 + |\nabla\beta|^2 - \frac{Q_{2m}(\beta)}{2[(2m-1)!]}\right]\eta . \tag{3.7}$$

By virtue of (3.5) and the relation $d\beta = 0$, a consequence of (3.2), from (3.7) it follows that

$$\frac{1}{2}\int_B (N, (d|\beta|^2))\bar{\eta} \geq \int_M [-|\delta\beta|^2 + 2mk|\beta|^2]\eta . \tag{3.8}$$

For a harmonic 2-form α tangential or normal to the boundary B, formula (2.5) becomes [10, pp. 435–436]

$$\frac{1}{2}(\Delta|\alpha|^2) = -|\nabla\alpha|^2 + \frac{1}{2}Q_2(\alpha) ,$$

from which we get

$$\frac{1}{2}|\alpha|^{2m-2}\Delta(|\alpha|^2) = -|\alpha|^{2m-2}|\nabla\alpha|^2 + \frac{1}{2}|\alpha|^{2m-2}Q_2(\alpha) ,$$

or

$$\frac{1}{2}\int_M |\alpha|^{2m-2}\Delta(|\alpha|^2)\eta = \int_M \left[-|\alpha|^{2m-2}|\nabla\alpha|^2 + \frac{1}{2}|\alpha|^{2m-2}Q_2(\alpha)\right]\eta ,$$

which together with (3.4) gives

$$\begin{aligned}-m\int_M |\alpha|^{2m-2}\Delta(|\alpha|^2)\eta \geq \int_M \Big[&2m|\alpha|^{2m-2}|\nabla\alpha|^2 + 4m(2m-1)k|\alpha|^{2m} \\ &- \frac{16}{3}m(1-k)\delta|\alpha|^{2m-2}\Big]\eta .\end{aligned} \tag{3.9}$$

It is well known that the following relation holds:

$$\Delta[(|\alpha|^2)^m] = m|\alpha|^{2m-2}\Delta(|\alpha|^2) - m(m-1)|\alpha|^{2m-4}(d(|\alpha|^2))^2 ,$$

which implies the inequality

$$\Delta[(|\alpha|^2)^m] \leq m|\alpha|^{2m-2}\Delta(|\alpha|^2) ,$$

or

$$\int_M \Delta[(|\alpha|^2)^m]\eta \leq \int_M m|\alpha|^{2m-2}\Delta(|\alpha|^2)\eta . \tag{3.10}$$

By means of (2.9), the relation (3.10) takes the form

$$\int_B (N, d[(|\alpha|^2)^m])\bar{\eta} \geq -\int_M m|\alpha|^{2m-2}\Delta(|\alpha|^2)\eta . \tag{3.11}$$

From (3.9) and (3.11) follows immediately the inequality

$$\begin{aligned}\frac{1}{2}\int_B (N, |\alpha|^{2m-2}d(|\alpha|^2))\bar{\eta} \geq \int_M \Big[&|\alpha|^{2m-2}|\nabla\alpha|^2 - \frac{8}{3}(1-k)\delta|\alpha|^{2m-2} \\ &+ 2(2m-1)k|\alpha|^{2m}\Big]\eta .\end{aligned} \tag{3.12}$$

By means of the inequality

$$|\delta\beta|^2 \leq \frac{(2m-1)(m-1)}{m^{m-2}}|\nabla\alpha|^2|\alpha|^{2m-2} \tag{3.13}$$

proved in [8], from (3.8) we obtain

$$\begin{aligned}\frac{m^{m-2}}{2(2m-1)(m-1)}\int_B (N, d(|\beta|^2))\bar{\eta} \geq \int_M \Big[&-|\nabla\alpha|^2|\alpha|^{2m-2} \\ &+ \frac{2m^{m-1}}{(2m-1)(m-1)}|\beta|^2\Big]\eta .\end{aligned} \tag{3.14}$$

Thus addition of (3.12) to (3.14) gives readily

$$\frac{1}{2}\int_B 3(N, m^{m-2}d(|\beta|^2) + (2m-1)(m-1)|\alpha|^{2m-2}d(|\alpha|^2))\bar{\eta}$$
$$\geq \int_M 2[3(2m-1)^2(m-1)k|\alpha|^{2m} - 4(2m-1)(m-1)(1-k)\delta|\alpha|^{2m-2}$$
$$+ 3m^{m-1}k|\beta|^2]\eta ,$$

or

$$\begin{aligned}(3.15)\qquad &\int_B 3[m^{m-2}(\nabla_l\beta_{hi_2\cdots i_{2m}})\beta^{hi_2\cdots i_{2m}}N^l, \qquad (h < i_2 < \cdots < i_{2m})\\ &\quad + (2m-1)(m-1)|\alpha|^{2m-2}(\nabla_l\alpha_{hi_2})\alpha^{hi_2}N^l, \qquad (h < i_2)]\bar{\eta}\\ &\geq \int_M 2[3(2m-1)^2(m-1)k|\alpha|^{2m} - 4(2m-1)(m-1)(k-1)\delta|\alpha|^{2m-2}\\ &\quad + 3km^{m-1}|\beta|^2]\eta .\end{aligned}$$

If the harmonic 2-form α is tangential to B, then by means of (2.10), $d\alpha = 0$ and $d\beta = 0$, (3.15) becomes

$$\begin{aligned}(3.16)\qquad &\int_B 3H_{ij}[2m^{m-1}\bar{\beta}^i{}_{i_2\cdots i_{2m}}\bar{\beta}^{ji_2\cdots i_{2m}}, \qquad (i_2 < \cdots < i_{2m})\\ &\quad + 2(2m-1)(m-1)|\alpha|^{2m-2}\bar{\alpha}^i_{i_2}\bar{\alpha}^{ji_2}]\bar{\eta}\\ &\geq \int_M 2[3(2m-1)^2(m-1)k|\alpha|^{2m} - 4(2m-1)(m-1)\delta|\alpha|^{2m-2}\\ &\quad + 3km^{m-1}|\beta|^2]\eta .\end{aligned}$$

We can prove with the same technique as in [8] that if k satisfies the inequality

$$(3.17)\qquad k > \lambda = 2(2m-1)(m-1)^2m/[m(m-1)(2m-1)(8m-5)+3] ,$$

then the second member of (3.16) is positive. Hence we have the following theorem and corollary.

Theorem I. *Let M be a compact k-pinched Riemannian manifold of dimension $n = 2m+1$ with boundary B. If $k > \lambda$, given by (3.17), and the second fundamental form on the boundary is semi-negative, then the second absolute Betti number A_2 of the manifold vanishes.*

Corollary I. *For a compact k-pinched Riemannian manifold M of dimension $n = 2m+1$ with a totally geodesic boundary, $A_2 = 0$ if*

$$k > 2(2m-1)(m-1)^2m/[m(m-1)(2m-1)(8m-5)+3] .$$

If the harmonic 2-form α is normal to B, then by means of (2.11) and $\delta\alpha = 0$, (3.15) takes the form

$$
\begin{aligned}
&\int_B 6\{m^{m-1}(\nabla_h \beta^h{}_{i_2\cdots i_{2m}})\beta^{l i_2\cdots i_{2m}} N_l \\
&\quad + H^l{}_l[m^{m-1}\overline{\beta}_{i_2\cdots i_{2m}}\overline{\beta}^{i_2\cdots i_{2m}} + (2m-1)(m-1)|\alpha|^{2m-2}\overline{\alpha}_{i_2}\overline{\alpha}^{i_2}] \\
&\quad - H_{ij}[(2m-1)m^{m-1}\overline{\beta}^i{}_{i_3\cdots i_{2m}}\overline{\beta}^{j i_3\cdots i_{2m}} \\
&\quad + (2m-1)(m-1)|\alpha|^{2m-2}\overline{\alpha}^i\overline{\alpha}^j]\}\overline{\eta}, \qquad (i_2 < i_3 < \cdots < i_{2m}) \\
&\geq \int_M 2[3(2m-1)^2(m-1)k|\alpha|^{2m} - 4(2m-1)(m-1)(k-1)\delta|\alpha|^{2m-2} \\
&\quad + 3km^{m-1}|\beta|^2]\eta\,.
\end{aligned} \tag{3.18}
$$

Denote the following quadratic form by $L(\alpha, \alpha)$:

$$
\begin{aligned}
L(\alpha,\alpha) &= m^{m-1}(\nabla_h \beta^h{}_{i_2\cdots i_{2m}})\beta^{l i_2\cdots i_{2m}} N^l \\
&\quad + H^l{}_l[m^{m-1}\overline{\beta}_{i_2\cdots i_{2m}}\overline{\beta}^{i_2\cdots i_{2m}} + (2m-1)(m-1)|\alpha|^{2m-2}\overline{\alpha}_{i_2}\overline{\alpha}^{i_2}] \\
&\quad - H_{ij}[(2m-1)m^{m-1}\overline{\beta}^i{}_{i_3\cdots i_{2m}}\overline{\beta}^{j i_3\cdots i_{2m}} \\
&\quad + (2m-1)(m-1)|\alpha|^{2m-2}\overline{\alpha}^i\overline{\alpha}^j]\,, \qquad (i_2 < i_3 < \cdots < i_{2m})\,.
\end{aligned} \tag{3.19}
$$

Therefore from (3.18) and (3.19) we conclude the theorem:

Theorem II. *Let M be a compact k-pinched Riemannian manifold with boundary B. If $k > \lambda$ given by* (3.17), *and the quadratic form $L(\alpha, \alpha)$ defined by* (3.19) *is semi-negative, then the second relative Betti number R_2 of the manifold* (*mod. B*) *is zero.*

4. In this section we use the same technique as in § 3 to improve the number λ if the dimension of the manifold is 5. By estimating the norm $|\delta\beta|^2$ at the point P and using the inequality

$$2(AB + CD)^2 \leq A^2(3B^2 + D^2) + C^2(B^2 + 3D^2)$$

we obtain

$$|\delta\beta|^2 \leq 5|\nabla\alpha|^2|\alpha|^2/2\,. \tag{4.1}$$

By means of (4.1) and $m = 2$, the inequality (3.8) becomes

$$\frac{1}{5}\int_B (N, d(|\beta|^2))\overline{\eta} \geq \int_M \left[-|\nabla\alpha|^2|\alpha|^2 + \frac{8}{5}k|\beta|^2\right]\eta\,. \tag{4.2}$$

Moreover for $m = 2$, (3.12) is reduced to

$$\frac{1}{2}\int_B (N, |\alpha|^2 d(|\alpha|^2))\overline{\eta} \geq \int_M \left[|\alpha|^2|\nabla\alpha|^2 - \frac{8}{3}(1-k)\delta|\alpha|^2 + 6k|\alpha|^4\right]\eta\,, \tag{4.3}$$

which together with (4.2) implies

$$\int_B (N, 6d(|\beta|^2) + 15|\alpha|^2 d(|\alpha|^2))\bar{\eta}$$
$$\geq \int_M [180k|\alpha|^4 - 80(1-k)\delta|\alpha|^2 + 48k|\beta|^2]\eta ,$$

or

$$(4.4) \quad \begin{aligned} &\int_B [6(\nabla_l \beta_{h i_2 i_3 i_4})\beta^{h i_2 i_3 i_4} N^l + 15|\alpha|^2(\nabla_l \alpha_{h i_2})\alpha^{h i_2} N^l]\bar{\eta}, \qquad (h < i_2 < i_3 < i_4) \\ &\geq \int_M [90k|\alpha|^4 - 40(1-k)\delta|\alpha|^2 + 24k|\beta|^2]\eta . \end{aligned}$$

If the harmonic 2-form α is tangential to B, then (4.4) takes the form

$$\int_B 3H_{ij}[8\bar{\beta}^i{}_{i_2 i_3 i_4}\bar{\beta}^{j i_2 i_3 i_4} + 10|\alpha|^2 \bar{\alpha}^i_{i_2}\bar{\alpha}^{j i_2}]\bar{\eta}, \qquad (i_2 < i_3 < i_4)$$
$$\geq \int_M [90k|\alpha|^4 - 40(1-k)\delta|\alpha|^2 + 24k|\beta|^2]\eta .$$

If $k > 10/58$, then the second member of the above inequality is positive. Thus we obtain the following theorem and corollary.

Theorem III. *Let M be a compact k-pinched Riemannian manifold of dimension 5 with boundary B. If $k > 10/58$ and the second fundamental form on B is semi-negative, then $A_2 = 0$.*

Corollary II. *For a compact k-pinched Riemannion manifold M of dimension 5 with a totally geodesic boundary, if $k > 10/58$, then $A_2 = 0$.*

If the harmonic 2-form α is normal to the boundary B, then (4.4) becomes

$$(4.5) \quad \begin{aligned} &\int_B 3[8(\nabla_h \beta^h{}_{i_2 i_3 i_4})\beta^{l i_2 i_3 i_4} N_l + H^l{}_l(8\bar{\beta}_{i_2 i_3 i_4}\bar{\beta}^{i_2 i_3 i_4} + 10|\alpha|^2 \bar{\alpha}_{i_2}\bar{\alpha}^{i_2}) \\ &\quad - H_{ij}(24\bar{\beta}^i{}_{i_3 i_4}\bar{\beta}^{j i_3 i_4} + 10|\alpha|^2 \bar{\alpha}^i \bar{\alpha}^j)]\bar{\eta}, \qquad (i_2 < i_3 < i_4) \\ &\geq \int_M [90k|\alpha|^4 - 40(1-k)\delta|\alpha|^2 + 24k|\beta|^2]\eta . \end{aligned}$$

From the relation (4.5) and the quadratic form

$$(4.6) \quad \begin{aligned} G(\alpha, \alpha) = {} & 8(\nabla_h \beta^h{}_{i_2 i_3 i_4})\beta^{l i_2 i_3 i_4} N_l + H^l{}_l(8\bar{\beta}_{i_2 i_3 i_4}\bar{\beta}^{i_2 i_3 i_4} + 10|\alpha|^2 \bar{\alpha}_{i_2}\bar{\alpha}^{i_2}) \\ & - H_{ij}(24\bar{\beta}^i{}_{i_3 i_4}\bar{\beta}^{j i_3 i_4} + 10|\alpha|^2 \bar{\alpha}^i \bar{\alpha}^j), \\ & (i_2 < i_3 < i_4) , \end{aligned}$$

we thus have

Theorem IV. *Let M be a compact k-pinched Riemannian manifold of dimension* 5 *with boundary B. If* $k > 10/58$ *and the quadratic form* $G(\alpha, \alpha)$ *defined by* (4.6) *is semi-negative, then the second relative Betti number* R_2 *of the manifold M (mod. B) vanishes.*

5. If the boundary $\partial M = B = \phi$ and the dimension of the manifold is 5, then from the relation (4.4) we obtain the following theorem and corollary:

Theorem V. *Let M be a compact orientable k-pinched Riemannian manifold of dimension* 5 *without boundary. If* $k > 10/58$, *then* $H^2(M, \mathbf{R}) = 0$.

Corollary III. *If a compact orientable k-pinched Riemannian manifold M of dimension* 5 *without boundary is homeomorphic to* $S^2 \times S^3$, *then* $k \leq 10/58$.

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